Advanced Microeconomics

Math You Should Know: Preliminaries

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Fall 2024

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2 Continuity and Compact Set

3 Concavity and Convexity

4 Homogeneous Functions and Euler's Formula

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Notations

- $\mathbb{R}^d = \underbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{d}$
- $\times$: Cartesian product

$$A \times B = \{(a, b) | a \in A \text{ and } b \in B\}$$

- $\|x\|$: Euclidean norm

$$\|x\| = \left(\sum_{i=1}^{d} x_i^2 \right)^{1/2}$$

- $\|x - y\|$: Euclidean distance

$$\|x - y\| = d(x, y) = \left(\sum_{i=1}^{d} (x_i - y_i)^2 \right)^{1/2}$$

Notations

- $f: \mathbb{R}^d \to \mathbb{R}^m$
- $\nabla f(x)$: gradient vector

$$\nabla f(x) = (f_1(x), f_2(x), \ldots, f_d(x)) = \left(\frac{\partial f(x)}{\partial x_1}, \frac{\partial f(x)}{\partial x_2}, \ldots, \frac{\partial f(x)}{\partial x_d} \right)$$

- $Df(x)$: the derivative of f at x

$$Df(x) = \begin{bmatrix} Df^1(x) \\ \vdots \\ Df^m(x) \end{bmatrix} = \begin{bmatrix} \nabla f^1(x) \\ \vdots \\ \nabla f^m(x) \end{bmatrix} \equiv \begin{bmatrix} \frac{\partial f^1(x)}{\partial x_1} & \cdots & \frac{\partial f^1(x)}{\partial x_d} \\ \vdots & \ddots & \vdots \\ \frac{\partial f^m(x)}{\partial x_1} & \cdots & \frac{\partial f^m(x)}{\partial x_d} \end{bmatrix}$$

  - wealth effect $D_w x(p, w)$
  - price effect $D_p x(p, w)$

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Compact Set

Compact Set

A set $\mathcal{D} \in \mathbb{R}^d$ is called compact if every sequence in $\mathcal{D}$ has a subsequence that converges to an element again contained in $\mathcal{D}$.

- (**Bolzano-Weierstrass Theorem**) A set $\mathcal{D} \in \mathbb{R}^d$ is compact if and only if it is closed and bounded.

Bounded Set

Bounded Set

A set $\mathcal{D} \in \mathbb{R}^d$ is bounded if there exists $k > 0$ such that $\|x\| < k$ for each $x \in \mathcal{D}$.

Example

- $\mathcal{D} = [0, 2]$
- $\mathcal{D} = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \leq 1\}$

Closed Set

Closed Set

A set $\mathcal{D} \in \mathbb{R}^d$ is closed if its complement $\mathcal{D}^C = \mathbb{R}^d \setminus \mathcal{D}$ is open.

- (**Equivalent Definition**) A set $\mathcal{D} \in \mathbb{R}^d$ is closed if and only if for every sequence $(x_n) \subset \mathcal{D}$ which converges in $\mathbb{R}^d$, the limit $\lim_{n\to\infty} = x$ must also lie in $\mathcal{D}$.

Example

- $\mathcal{D} = [0, 1]$
- $\mathcal{D} = [0, 1)$

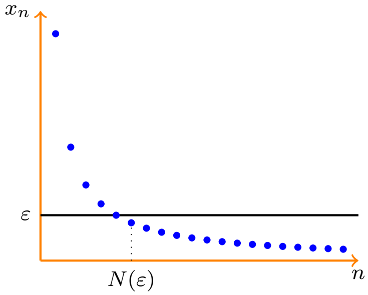
Convergence

Convergence

A sequence (x_n) in $\mathbb{R}^d$ is said to converge to a limit x (written $x_n \to x$) if for all $\varepsilon > 0$, there exists an integer $N(\varepsilon)$ such that for all $n \geq N(\varepsilon)$, we have $d(x_n, x) < \varepsilon$.

Example

- Consider the sequence (x_n), where $x_n = \frac{1}{n}, n = 1, 2, \ldots$.



Continuous Function

Continuous Function

A function $f : \mathcal{D} \to \mathcal{T}$ where $\mathcal{D} \subset \mathbb{R}^d$ and $\mathcal{T} \subset \mathbb{R}^m$ is continuous at $x \in \mathcal{D}$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x \in \mathcal{D}$,

$$d(x, y) < \delta \quad \Longrightarrow \quad d(f(x), f(y)) < \varepsilon$$

- (**Equivalent Definition**) f is continuous at $x \in \mathcal{D}$ if for all (x_n) such that $x_n \in \mathcal{D}$ for each n and $x_n \to x$, we have $f(x_n) \to f(x)$.

$$\forall (x_n)_{n \in \mathbb{N}} \subset \mathcal{D} : \lim_{n \to \infty} x_n = x \Longrightarrow \lim_{n \to \infty} f(x_n) = f(x)$$

f is continuous on $\mathcal{D}$ if it is continuous at each $x \in \mathcal{D}$.

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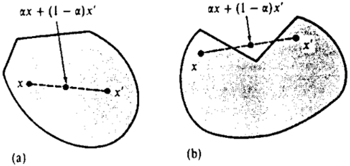
Convex Set and Separating Hyperplane

Definition

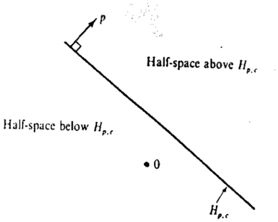
A set $A \subset \mathbb{R}^n$ is convex if $\alpha x + (1-\alpha)x' \in A$ whenever $x, x' \in A$ and $\alpha \in [0,1]$.

Definition

Given $p \in \mathbb{R}^n$ with $p \neq 0$, and $c \in \mathbb{R}$. The **hyperplane** generated by p and c is the set $H_{p,c} = \{z \in \mathbb{R}^n : p \cdot z = c\}$. The sets $H_{p,c} = \{z \in \mathbb{R}^n : p \cdot z \geq c\}$ and $H_{p,c} = \{z \in \mathbb{R}^n : p \leq z = c\}$ are called the half-space above and the half-space below $H_{p,c}$, respectively.



Convex set and nonconvex set

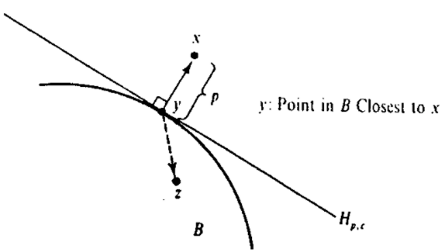


Hyperplane and half-spaces

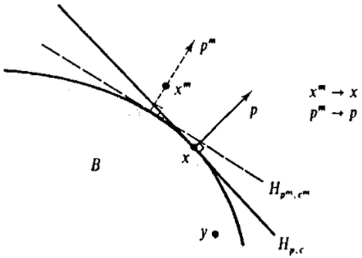
Separating Hyperplane and Supporting Hyperplane

Theorem

- (**Separating Hyperplane Theorem**) Assume that $B \subset \mathbb{R}^n$ is convex and closed, and that $x \notin B$. Then there is $p \in \mathbb{R}^n$ with $p \neq 0$, and a value $c \in \mathbb{R}$ such that $p \cdot x > c$ and $p \cdot y < c$ for every $y \in B$.
- (**Supporting Hyperplane Theorem**) Assume that $B \subset \mathbb{R}^n$ is convex, and that x is not an element of the interior of B. Then there is $p \in \mathbb{R}^n$ with $p \neq 0$, such that $p \cdot x \geq p \cdot y$ for every $y \in B$.



Separating Hyperplane Theorem



Supporting Hyperplane Theorem

Concavity v.s Convexity

Let $\mathcal{D} \subset \mathbb{R}^n$ and $f : \mathcal{D} \to \mathbb{R}$. From now on we assume $\mathcal{D}$ is convex.

Definition

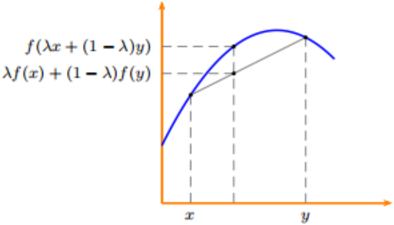
A function $f : \mathcal{D} \to \mathbb{R}$ is concave if for any $x, y \in \mathcal{D}$ and $\lambda \in (0, 1)$, it is the case that

$$f(\lambda x + (1 - \lambda)y) \geq \lambda f(x) + (1 - \lambda)f(y)$$

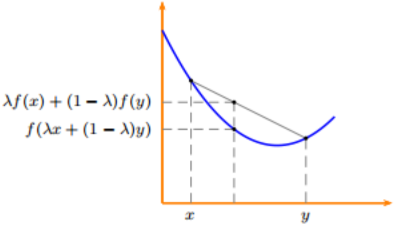
and convex if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

Concavity v.s Convexity



A concave function



A convex function

Concavity v.s Convexity

Definition

A function $f : \mathcal{D} \to \mathbb{R}$ is strictly concave if for any $x \neq y \in \mathcal{D}$ and $\lambda \in (0, 1)$, it is the case that

$$f(\lambda x + (1 - \lambda)y) > \lambda f(x) + (1 - \lambda)f(y)$$

and strictly convex if

$$f(\lambda x + (1 - \lambda)y) < \lambda f(x) + (1 - \lambda)f(y)$$

Theorem

- (**negative of a function**)$f : \mathcal{D} \to \mathbb{R}$ is (strictly) concave if and only if $-f$ is (strictly) convex.
- (**sum of functions**) If $f(x)$ and $g(x)$ are both concave (convex) functions, then $f(x) + g(x)$ is also a concave (conex) function.

Convex Set v.s Convex Function

1. In defining a convex function, we need a convex set for the domain.
2. If $f(x)$ is a convex function, then for any constant k, it can give rise to a convex set $S^{\leq} \equiv \{x | f(x) \leq k\}$:

$$f(x) \text{ is convex} \quad \Longrightarrow \quad S^{\leq} \text{ is convex}$$

   If $f(x)$ is a concave function, then for any constant k, it can give rise to a convex set $S^{\geq} \equiv \{x | f(x) \geq k\}$

$$f(x) \text{ is concave} \quad \Longrightarrow \quad S^{\geq} \text{ is convex}$$

Quasi-concavity v.s. Quasi-convexity

Let $\mathcal{D} \subset \mathbb{R}^n$ and $f : \mathcal{D} \to \mathbb{R}$. From now on we assume $\mathcal{D}$ is convex.

Definition

A function $f : \mathcal{D} \to \mathbb{R}$ is quasi-concave if for any $x, y \in \mathcal{D}$ and $\lambda \in (0, 1)$, it is the case that

$$f(\lambda x + (1 - \lambda)y) \geq \min\{f(x), f(y)\}$$

and quasi-convex if

$$f(\lambda x + (1 - \lambda)y) \leq \max\{f(x), f(y)\}$$

Quasi-concavity v.s. Quasi-convexity

Let $\mathcal{D} \subset \mathbb{R}^n$ and $f : \mathcal{D} \to \mathbb{R}$. From now on we assume $\mathcal{D}$ is convex.

Definition

A function $f : \mathcal{D} \to \mathbb{R}$ is strictly quasi-concave if for any $x, y \in \mathcal{D}$ and $\lambda \in (0, 1)$, it is the case that

$$f(\lambda x + (1-\lambda)y) > \min\{f(x), f(y)\}$$

and strictly quasi-convex if

$$f(\lambda x + (1-\lambda)y) < \max\{f(x), f(y)\}$$

Theorem

- (**negative of a function**) If $f(x)$ is (strictly) quasi-concave, then $-f(x)$ is (strictly) quasi-convex.
- (**concavity v.s quasi-concavity**) (strictly) concavity $\Rightarrow$ (strictly) quasi-concavity; (strictly) convexity $\Rightarrow$ (strictly) quasi-convexity.

Convex Set, Convex Function and Quasi-Convex Function

Theorem

- Convex Function $\Longrightarrow$ Quasi-Convex Function $\Longleftrightarrow S^{\leq} \equiv \{x|f(x) \leq k\}$ is convex.
- Concave Function $\Longrightarrow$ Quasi-Concave Function $\Longleftrightarrow S^{\geq} \equiv \{x|f(x) \geq k\}$ is convex.

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Homogeneous Degree

Definition

A function $f(x_1, \ldots, x_N)$ is **homogeneous of degree** r (for $r = \ldots, -1, 0, 1, \ldots$) if for every $t > 0$ we have

$$f(tx_1, \ldots, tx_N) = t^r f(x_1, \ldots, x_N)$$

Example

- $f(x_1, x_2) = x_1/x_2$ is HD0.
- $f(x_1, x_2) = (x_1 x_2)^{1/2}$ is HD1.

Theorem

If $f(x_1, \ldots, x_N)$ is homogeneous of degree r (for $r = \ldots, -1, 0, 1, \ldots$), then for any $n = 1, \cdots, N$ the partial derivative function $\partial f(x_1, \ldots, x_N)/\partial x_n$ is homogeneous of degree $r - 1$.

Euler's Formula

Euler's Formula

If $f(x_1, \ldots, x_N)$ is homogeneous of degree r (for $r = \ldots, -1, 0, 1, \ldots$) and differentiable, then at any point we have

$$\sum_{n=1}^{N} \frac{\partial f\left(\bar{x}_1, \ldots, \bar{x}_N\right)}{\partial x_n} \tilde{x}_n = r f\left(\bar{x}_1, \ldots, \bar{x}_N\right)$$

Proof.

By definition we have

$$f\left(t x_1, \ldots, t x_N\right) = t^r f\left(x_1, \ldots, x_N\right)$$

Differentiating this equation w.r.t. t gives

$$\sum_{n=1}^{N} \frac{\partial f\left(t\bar{x}_1, \ldots, t\bar{x}_N\right)}{\partial x_n} \bar{x}_n - r t^{r-1} f\left(\bar{x}_1, \ldots, \bar{x}_N\right) = 0$$

Let $t = 1$ and we obtain the Euler's Rule. □

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Optimization Problem

- Optimization means finding the maximum or minimum values of a quantity, and finding when these max/mins occurs.
- What quantities are optimized in economics?
  - maximize utility (UMP), profit (PMP) ...
  - minimize expenditure (EMP), cost (CMP) ...
- 1^{st} steps to solve an Optimization Problem
  - Identify the quantity to be optimized
  - Identify the feasible domain

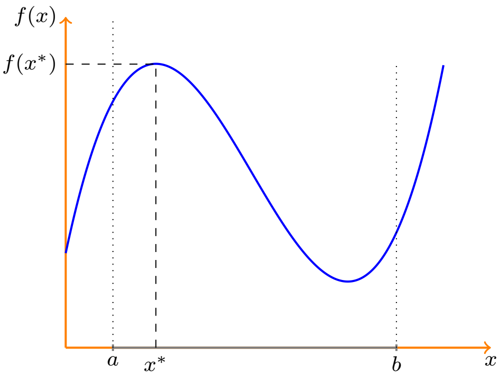
Optimization Problem

Let $f : \mathbb{R}^d \to \mathbb{R}$ and $\mathcal{D} \subset \mathbb{R}^d$. A constrained optimization problem is

$$\max f(x) \qquad \text{subject to } x \in \mathcal{D}$$

- f is the objective function
- $\mathcal{D}$ is the constraint set
- A solution to this problem $x \in \mathcal{D}$ is called a maximizer
- The set of maximizers is denoted
$$\text{argmax}\{f(x) | x \in \mathcal{D}\}$$
- Similarly for minimization problems

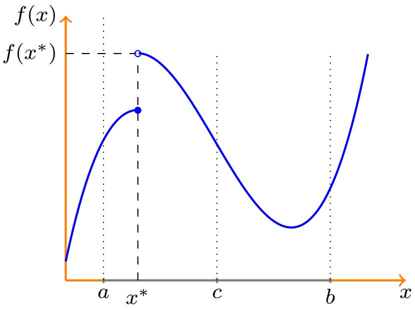
A Graphical Example



Existence of Solution

Let $f : \mathbb{R} \to \mathbb{R}$ and consider the following problems

- $\max f(x) \quad \text{s.t. } x \in [a, b]$
- $\max f(x) \quad \text{s.t. } x \in [c, b]$
- $\max f(x) \quad \text{s.t. } x \in (c, b)$



Existence of Solutions

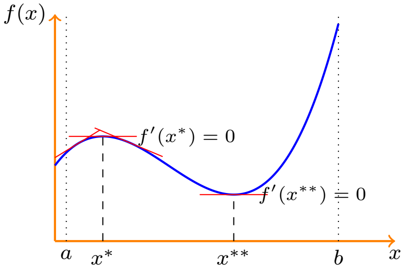
Theorem (Weierstrass Theorem)

Let $\mathcal{D} \in \mathbb{R}^d$ be compact and $f : \mathcal{D} \to \mathbb{R}$ be a continuous function on $\mathcal{D}$. Then f attains a maximum and a minimum on $\mathcal{D}$.

- Compact Set
  - Bounded Set
  - Closed Set
    - Convergence
- Continuous Function

A Simple Case

Let $f : \mathbb{R} \to \mathbb{R}$ and consider the problem $\max_{x \in [a,b]} f(x)$.



Theorem

Let $f : \mathbb{R} \to \mathbb{R}$ and suppose $a < x^* < b$ is a local maximum (minimum) of f on $[a, b]$. Then, $f'(x^*) = 0$.

- We call a point x^* such that $f'(x^*) = 0$ a critical point.

Recipe for solving the simple case

Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function and consider the problem $\max_{x \in [a,b]} f(x)$. If the problem has a solution, then it can be found by the following method:

1. Find all critical points: i.e., $x^* \in [a, b]$ s.t. $f'(x^*) = 0$
2. Evaluate f at all critical points and at boundaries a and b
3. The one that gives the highest f is the solution

- Note that if $f'(a) > 0$ (or $f'(b) < 0$), then the solution cannot be at a (or b)

Recipe for general problems

Generalizes to $f : \mathbb{R}^d \to \mathbb{R}$ and the problem is

$$\max f(x) \quad \text{subject to } x \in \mathcal{D}$$

1. Find critical points $x^* \in \mathcal{D}$ such that $Df(x^*) = 0$
2. Evaluate f at the critical points and the boundaries of $\mathcal{D}$
3. Choose the one that give the highest f

- Important to remember that solution must exist for this method to work
- In more complicated problems evaluating f at the boundaries could be difficult
- For such cases we have the method of the Lagrangean (for equality constraints) and Kuhn-Tucker conditions (for inequality constraints)

Equality Constraints: Lagrangean Method

Let $f : \mathbb{R}^d \to \mathbb{R}$ and $g_j : \mathbb{R}^d \to \mathbb{R}$ for each $j = 1, \dots, k$ be $\mathcal{C}^1$ and consider the problem

$$\max f(x) \quad \text{s.t. } g_j(x) = 0, \qquad j = 1, \dots, k$$

1. Form the Lagrangean (λ_js are called Lagrangean multipliers)

$$L(x, \lambda) = f(x) + \sum_{j=1}^{k} \lambda_j g_j(x)$$

2. Find critical points of $L(x, \lambda)$

$$\frac{\partial L}{\partial x_i}(x, \lambda) = 0, \qquad i = 1, \dots, d$$
$$\frac{\partial L}{\partial \lambda_j}(x, \lambda) = 0, \qquad j = 1, \dots, k$$

3. Evaluate f at each critical point x, choose the one that gives the highest value

An Interpretation of the Lagrange Multiplier

Consider the problem

$$\max f(x) \quad \text{s.t. } g(x) = c$$

We can write the Lagrangian function as

$$L(x, \lambda) = f(x) + \lambda[c - g(x)].$$

For stationary values of $L(x, \lambda)$, regarded as a function of x and λ, the FOC is

$$\begin{aligned} L_\lambda &= c - g(x) = 0 \\ L_x &= \nabla f(x)^T - \lambda \nabla g(x)^T = 0, \end{aligned}$$

from which we can find the critical values $x^*(c)$ and $\lambda^*(c)$.

An Interpretation of the Lagrange Multiplier

Then we have the equilibrium identities

$$c - g(x^*) \equiv 0; \qquad \nabla f(x^*)^T - \lambda \nabla g(x^*)^T \equiv 0$$

Now since the optimal value of $L(x, \lambda)$ depends on λ^* and x^*, that is

$$L^* = f(x^*) + \lambda^*[c - g(x^*)].$$

Differentiating L^* totally with respect to c, we find

$$\begin{aligned}
\frac{dL^*}{dc} &= \nabla f(x^*) \cdot \nabla x^*(c)^T + [c - g(x^*)]\frac{d\lambda^*}{dc} + \lambda^* \left[1 - \nabla g(x^*) \cdot \nabla x^*(c)^T\right] \\
&= [\nabla f(x^*) - \lambda^* \nabla g(x^*)] \cdot \nabla x^*(c)^T + [c - g(x^*)]\frac{d\lambda^*}{dc} + \lambda^* \\
&= \lambda^*
\end{aligned}$$

Thus λ^* measures the sensitivity of Z^* to changes in the constraint.

Inequality Constraints: Kuhn-Tucker Method

Let $f : \mathbb{R}^n \to \mathbb{R}$ and $g_j : \mathbb{R}^n \to \mathbb{R}$ for each $j = 1, \ldots, k$ be $\mathcal{C}^1$ and consider the problem

$$\max f(x) \quad \text{s.t. } g_j(x) \le 0, \quad j = 1, \ldots, k$$

1. Form the Lagrangean (λ_js are called Lagrange multipliers)

$$L(x, \lambda) = f(x) + \sum_{j=1}^{k} \lambda_j g_j(x)$$

2. Applying Kuhn-Tucker (necessary) conditions to derive x^*

$$x_i \frac{\partial L}{\partial x_i} = 0, \quad i = 1, \ldots, d$$

$$\frac{\partial L}{\partial \lambda_j} \ge 0 \quad \lambda_j \ge 0 \quad \text{and} \quad \lambda_j \frac{\partial L}{\partial \lambda_j} = 0, \quad j = 1, \ldots, k$$

Interpretation of the K-T Conditions

$\partial L/\partial\lambda \geq 0 \quad \lambda \geq 0 \quad \text{and} \quad \lambda(\partial L/\partial\lambda) = 0$

- $\partial L/\partial\lambda^* = 0$ and $\lambda^* > 0$
  If the shadow price of wealth is positive in the optimal solution ($\lambda^* > 0$), then it is fully expended ($\partial L/\partial\lambda^* = 0$)
- $\partial L/\partial\lambda^* > 0$ and $\lambda^* = 0$
  If the consumer's wealth is not fully expended in the optimal solution ($\partial L/\partial\lambda^* > 0$), the shadow price of wealth must be set equal to zero ($\lambda^* = 0$).

Kuhn-Tucker Sufficiency

Kuhn-Tucker Sufficiency

Suppose f is concave and each g_j is convex. If x satisfies the K-T conditions, then x is a global solution to the constrained optimization problem.

We can **weaken** the conditions in the above theorem **when there is only one constraint**. Let $\mathcal{D} = \{x \in \mathbb{R}^n : g(x) \leq d\}$.

Kuhn-Tucker Sufficiency (for one constraint)

Suppose f is quasi-concave and g is convex. If x satisfies the K-T, then x is a global solution to the constrained optimization problem.

Uniqueness

Uniqueness

Suppose f is strictly quasi-concave and g is convex. If x satisfies the K-T, then x is the only global solution to the constrained optimization problem.

Proof.

We can prove this proposition by contradiction.

Assume x satisfies the K-T but is not the only solution to the problem, which means there exists $x' \neq x$ satisfying $f(x') = f(x)$ and $g(x') \leq d$.

Consider $x^* = \lambda x + (1 - \lambda)x'$ where $\lambda \in (0, 1)$.

f is strictly quasi-concave $\Rightarrow f(x^*) > f(x)$

g is convex $\Rightarrow g(x^*) \leq \lambda g(x) + (1 - \lambda)g(x') \leq d \Rightarrow x^* \in \mathcal{D}$

It is contradictory to the fact that x is a global solution. □

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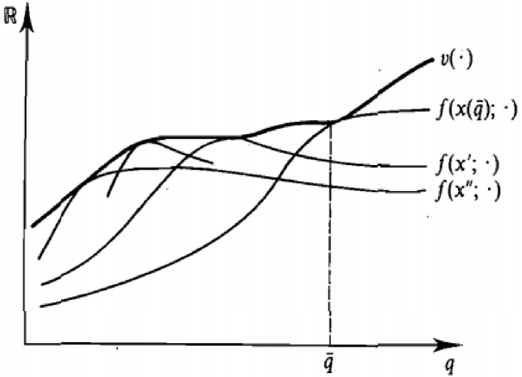
The Envelope Theorem

Consider the problem we discussed before

$$\max f(x) \quad \text{s.t. } g(x) = c$$

We denote the **value function** by $v(\cdot)$, which means $v(q)$ is the maximum value attained by $f(\cdot)$ when the parameter vector is q. The envelop theorem addresses the marginal effects of change in q on the value $v(q)$.



The Envelope Theorem

Consider the simplest case: one variable, one parameter, and no constraints. By the chain rule, we have

$$\frac{dv(\bar{q})}{dq} = \frac{\partial f(x(\bar{q});\bar{q})}{\partial q} + \frac{\partial f(x(\bar{q});\bar{q})}{\partial x}\frac{dx(\bar{q})}{dq}$$

According to the first-order conditions, we know $\frac{\partial f(x(\bar{q});\bar{q})}{\partial x} = 0$. Thus, the above equation simplifies to

$$\frac{dv(\bar{q})}{dq} = \frac{\partial f(x(\bar{q});\bar{q})}{\partial q}$$

Notice the only effect is the direct effect. Then we apply the conclusion to general cases.

The Envelope Theorem

Consider the value function $v(q)$ for the problem at last page. Assume that it is differentiable at $\bar{q} \in \mathbb{R}^d$ and that $(\lambda_1, \ldots, \lambda_M)$ are values of Lagrange multipliers associated with the maximizer solution $x(\bar{q})$ at $\bar{q}$. Then,

$$\frac{\partial v(\bar{q})}{\partial q_s} = \frac{\partial f(x(\bar{q});\bar{q})}{\partial q_s} - \sum_{m=1}^{M} \lambda_m \frac{\partial g_m(x(\bar{q});\bar{q})}{\partial q_s} \quad \text{for } s = 1, \ldots, S$$